(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

Prava i ravan

Prava $a: \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$, $\vec{p} = (l, m, n)$

Ravan $\alpha: Ax + By + Cz + D = 0$, $\vec{n} = (A, B, C)$

1° Ugao između prave a i ravni α $\quad \sin\varphi = \frac{\vec{p} \cdot \vec{n}}{|\vec{p}| \cdot |\vec{n}|}$

uslov paralelnosti $Al + Bm + Cn = 0$
$(\vec{p} \perp \vec{n})$

uslov normalnosti $\frac{A}{l} = \frac{B}{m} = \frac{C}{n}$ $\quad (\vec{p} \parallel \vec{n})$

2° Tačka prodora prave i ravni nalazi se tako što se napišu parametarske jednačine prave $x = x_1 + lt$, $y = y_1 + mt$, $z = z_1 + nt$ i zamjene vrijednosti x, y, z u jednačini ravni. Iz tako dobijene jednačine odredi se parametar t a samim tim i koordinate prodora.

3° Uslov da prava a leži u ravni α:

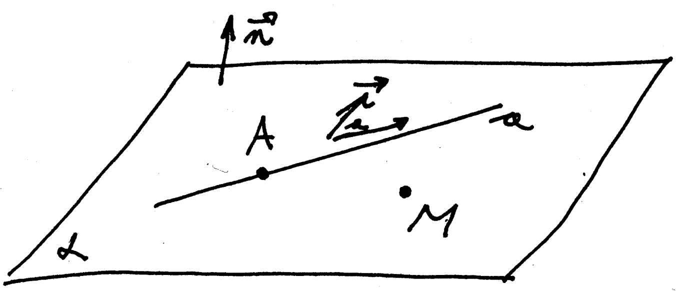
a) $Ax_1 + By_1 + Cz_1 + D = 0$ $\quad$ ($M_1(x_1, y_1, z_1)$ tačka na pravoj a),

b) $Al + Bm + Cn = 0$

(#) Napisati jednačinu ravni koja sadrži datu tačku $M(4, 5, 0)$ i datu pravu $\frac{x+3}{5} = \frac{y-4}{-3} = \frac{z-2}{2}$.

Rj.

$a: \frac{x+3}{5} = \frac{y-4}{-3} = \frac{z-2}{2}$



$A \in a \quad A(-3, 4, 2)$

$\vec{p}_a \{5, -3, 2\}$

$\alpha = ?$ $\quad \alpha: A(x-x_1) + B(y-y_1) + C(z-z_1)$

$\vec{n}\{A, B, C\}$

$A(-3,4,2)$
$M(4,5,0)$ $\Rightarrow \overrightarrow{AM}\{7,1,-2\}$

$$\left.\begin{array}{l}\vec{n}\perp\vec{p}_a\\ \vec{n}\perp\overrightarrow{AM}\end{array}\right\}\Rightarrow \vec{n}\parallel \vec{p}_a\times\overrightarrow{AM}$$

$$\Downarrow$$

$$\vec{n}=k(\vec{p}_a\times\overrightarrow{AM})$$

$k\in R$

$$\vec{p}_a\times\overrightarrow{AM}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 5 & -3 & 2\\ 7 & 1 & -2\end{vmatrix}=\vec{i}(6-2)-\vec{j}(-10-14)+\vec{k}(5+21)$$

$$=4\vec{i}+24\vec{j}+26k=\{4,24,26\}$$

$$=2\{2,12,13\}$$

Pa mogu uzeti $\vec{n}\{2,12,13\}$ (A B C)

α: $2(x-4)+12(y-5)+13(z-0)=0$

$2x+12y+13z-8-60=0$

$2x+12y+13z-68=0$ jednačina tražene ravni

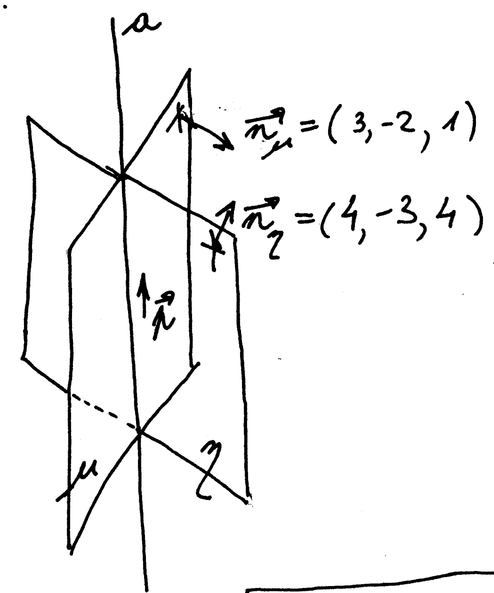
(#) Nađi konstante α i β tako da prava

$$a:\begin{cases}3x-2y+z+3=0\\ 4x-3y+4z+1=0\end{cases}$$

bude okomita na ravan

$$\alpha x+8y+\beta z+2=0.$$

Rj.



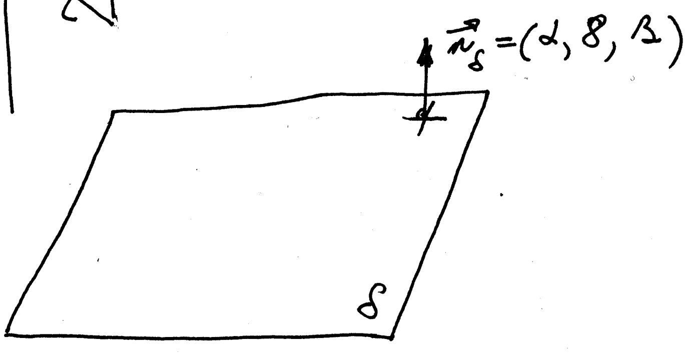
$\mu: 3x-2y+z+3=0$
$\eta: 4x-3y+4z+1=0$
$\delta: \alpha x+8y+\beta z+2=0$

$\mu\cap\eta=a$

$$\left.\begin{array}{l}\vec{p}\perp\vec{n}_\mu\\ \vec{p}\perp\vec{n}_\eta\end{array}\right.\Rightarrow \vec{p}\parallel\vec{n}_\mu\times\vec{n}_\eta$$

$$\Downarrow$$

$\exists k\in R$ $\vec{p}=k(\vec{n}_\mu\times\vec{n}_\eta)$



$$\vec{n}_\mu\times\vec{n}_\eta=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & -2 & 1\\ 4 & -3 & 4\end{vmatrix}=$$

$$=-5\vec{i}-8\vec{j}-\vec{k}=(-5,-8,-1)$$

$$\vec{p}=k(-5,-8,-1)$$

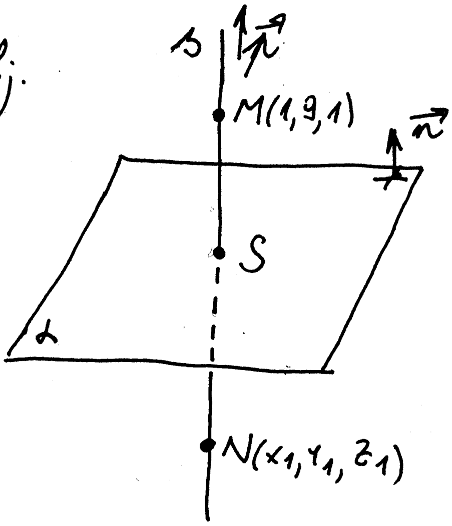
$\vec{p}\parallel\vec{n}_\delta \Rightarrow \exists s\in R: \vec{n}_\delta=s\cdot\vec{p}$

prema tome:
$\vec{n}_\delta=s\cdot k(-5,-8,-1)$

$\vec{n}_\delta=(5,8,1)$ $\Rightarrow$ $\alpha=5$, $\beta=1$ tražene vrijednosti

Odrediti tačku koja je simetrična tački $M(1,9,1)$ u odnosu na ravan $\alpha: 2x+y+3z=0$.

Rj.



$M(1,9,1)$

$\alpha: 2x+y+3z=0$

$M \notin \alpha$

$N=?$ $\qquad |\overrightarrow{MS}| = |\overrightarrow{NS}|$

Da bismo odredili tačku N prvo ćemo postaviti pravu s koja je okomita na α i uz pomoć te prave naći tačku S.

$\vec{n}=(2,1,3)$

$\vec{p} \parallel \vec{n} \Rightarrow$ mogu uzeti $\vec{p}=(2,1,3)$ $\qquad s: \dfrac{x-1}{2} = \dfrac{y-9}{1} = \dfrac{z-1}{3}$ (=t

$$s: \begin{cases} x=2t+1 \\ y=t+9 \\ z=3t+1 \end{cases} \qquad \begin{aligned} x-1&=2t \\ y-9&=t \\ z-1&=3t \end{aligned}$$

$2x+y+3z=0$

$2(2t+1)+(t+9)+3(3t+1)=0$

$4t+2+t+9+9t+3=0$

$14t=-14$

$t=-1$

Tačka presjeka prave s i ravni α je $S(-1,8,-2)$

$M(1,9,1)$

$S(-1,8,-2)$ $\qquad \overrightarrow{MS}=(-2,-1,-3)$

$N(2t+1, t+9, 3t+1)$

$S(-1, 8, -2)$ $\qquad \overrightarrow{NS}=(-2t-2, -t-1, -3t-3)$

$|\overrightarrow{MS}| = \sqrt{4+1+9} = \sqrt{14}$

$|\overrightarrow{NS}| = \sqrt{(-2t-2)^2+(-t-1)^2+(-3t-3)^2}$

$(-2t-2)^2 = 4t^2+8t+4$

$(-t+1) = t^2+2t+1$

$(-3t-3)^2 = 9t^2+18t+9$

$14t^2+28t+14$

$|\overrightarrow{MS}| = |\overrightarrow{NS}|$

$14t^2+28t+14 = 14 \quad /:14$

$t^2+2t=0$

$t(t+2)=0$

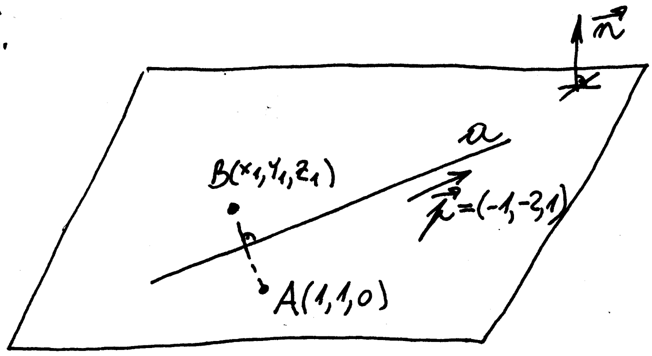
$t=0$ ili $t=-2$

$N(-3,7,-5)$

tražena tačka

(#) Data je prava $a: \frac{x+1}{-1} = \frac{y-2}{-2} = \frac{z}{1}$ i tačka A $A(1,1,0)$. Nađi jednačinu ravni koja sadrži pravu a i tačku A i tačku B simetričnu tački A u odnosu na pravu a.

Rj.



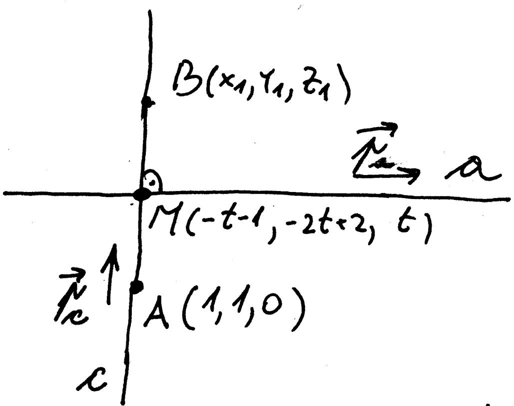
Nađimo prvo tačku $B(x_1,y_1,z_1)$.

$$a: \frac{x+1}{-1} = \frac{y-2}{-2} = \frac{z}{1} \quad (=t)$$

$$a: \begin{cases} x = -t-1 \\ y = -2t+2 \\ z = t \end{cases}$$

Da bi našao tačku B prvo trebamo naći pravu koja prolazi kroz tačke A i B.

$M(-t-1, -2t+2, t)$



$$\vec{AM} = (-t-2, -2t+1, t)$$

$$\vec{p} \perp \vec{AM}$$

$$\vec{p} \cdot \vec{AM} = 0 \quad \text{tj.} \quad (-1,-2,1)\cdot(-t-2,-2t+1,t) = 0$$

$$t+2+4t-2+t = 0$$

$$6t = 0 \Rightarrow t = 0$$

$$M(-1, 2, 0)$$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$ jednačina prave kroz dvije tačke

$$c: \frac{x-1}{-2} = \frac{y-1}{1} = \frac{z}{0}, \quad \vec{p}_c = (-2,1,0)$$

Napisaćemo jednačinu ravni koja sadrži pravu a i pravu c (kako ravan sadrži pravu c time će sadržati i tačku B)

$$\left.\begin{array}{l} \vec{n} \perp \vec{p} \\ \vec{n} \perp \vec{p}_c \end{array}\right\} \Rightarrow \vec{n} \parallel \vec{p} \times \vec{p}_c \Rightarrow k \in R: \vec{n} = k\cdot(\vec{p}\times\vec{p}_c)$$

$$\vec{p}\times\vec{p}_c = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -2 & 1 \\ -2 & 1 & 0 \end{vmatrix} = -\vec{i} - 2\vec{j} - 5\vec{k} = (-1,-2,-5) \Rightarrow \vec{n} = (1,2,5)$$

$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$, jednačina ravni

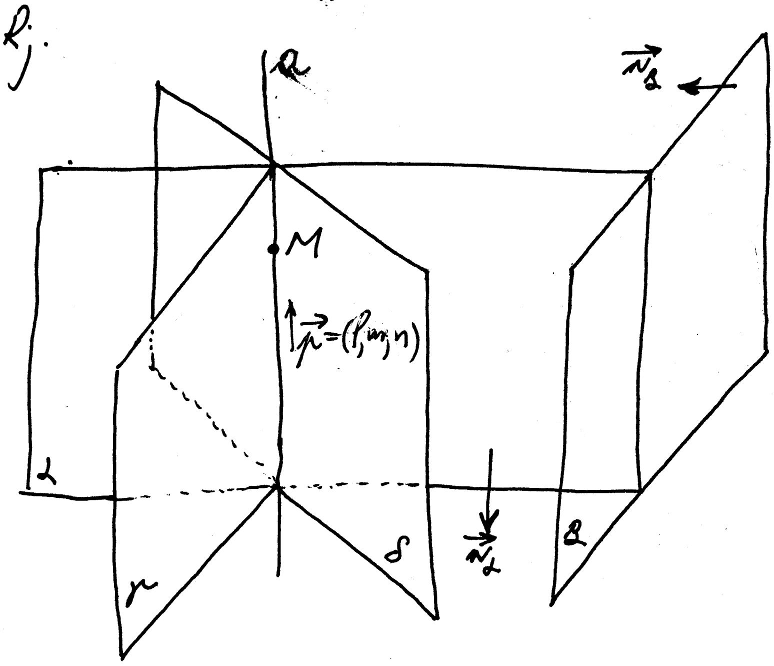
$$1(x-1)+2(y-1)+5(z-0)=0$$

$x+2y+5z-3=0$ jednačina tražene ravni

Napisati jednačinu ravni koja prolazi kroz presjek ravni $\begin{cases} x-y+z+1=0 \\ x+y-z+1=0 \end{cases}$

a normalna je na ravan $2x-y+5z-3=0$.

Rj.



$\alpha: A(x-x_1)+B(y-y_1)+C(z-z_1)=0$

$\beta: 2x-y+5z-3=0$

pramen ravni:

$A_1x+B_1y+C_1z+D_1+$

$+\lambda(A_2x+B_2y+C_2z+D_2)=0$

gdje su

$A_1x+B_1y+C_1z+D_1=0$ i

$A_2x+B_2y+C_2z+D_2=0$

dvije neparalelne ravni koje se sijeku po pravoj.

$x-y+z+1+\lambda(x+y-z+1)=0$

$x+\lambda x-y+\lambda y+z-\lambda z+1+\lambda=0$

$x(1+\lambda)+y(-1+\lambda)+z(1-\lambda)+(1+\lambda)=0$ pramen ravni koje prolaze kroz pravu a

$\vec{n}_\alpha=(1+\lambda, -1+\lambda, 1-\lambda)$

$\vec{n}_\beta=(2,-1,5)$

$\vec{n}_\alpha \perp \vec{n}_\beta \Rightarrow \vec{n}_\alpha \cdot \vec{n}_\beta=0$

$(1+\lambda, -1+\lambda, 1-\lambda)(2,-1,5)=0$

$2+2\lambda+1-\lambda+5-5\lambda=0$

$-4\lambda+8=0$

$\lambda=2$

$\vec{n}_\alpha=(3,1,-1)$

Treba nam još tačka $M\in a$

$a=\begin{cases} x-y+z+1=0 \\ x+y-z+1=0 \end{cases}$ $(M\in\gamma\wedge\delta)$

$2x+2=0$

$x=-1$

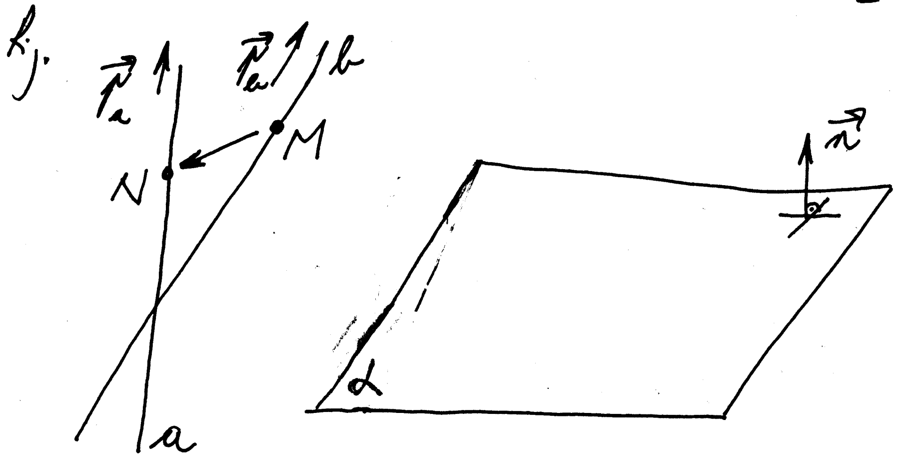
$M(-1,0,0)$

$3(x+1)+1(y-0)-1(z-0)=0$

$3x+y-z+3=0$ jednačina tražene ravni

(#) Napisati jednačinu prave koja prolazi kroz tačku $M(3,-2,-4)$, paralelna je ravni $\alpha: 3x-2y-3z-7=0$ i siječe pravu $a: \frac{x-2}{3} = \frac{y+4}{-2} = \frac{z-1}{2}$.

Rj.



$b: \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

$b = ?$ jednačina prave

$M(3,-2,-4)$, $\vec{n}=(3,-2,-3)$

$\vec{p}_a = (3,-2,2)$

$N \in a$, $N(2,-4,1)$

$\overrightarrow{MN} = (-1,-2,-3)$

Vektor $\vec{p}_a$, $\overrightarrow{MN}$ i $\vec{p}_b$ leže u istoj ravni, pa imamo:

$(\vec{p}_a \times \vec{p}_b)\cdot \overrightarrow{MN} = 0$ tj. $\begin{vmatrix} 3 & -2 & 2 \\ l & m & n \\ -1 & -2 & -3 \end{vmatrix} = 0$

$\vec{p}_b \perp \vec{n} \Rightarrow \vec{p}_b \cdot \vec{n} = 0$

$(l,m,n)\cdot(3,-2,-3) = 0 \Rightarrow 3l-2m-3n=0$

$$\begin{vmatrix} 3 & -2 & 2 \\ l & m & n \\ -1 & -2 & -3 \end{vmatrix} = (-1)\begin{vmatrix} 3 & -2 & 2 \\ l & m & n \\ 1 & 2 & 3 \end{vmatrix} \frac{II_k - I_k\cdot 2}{III_k - I_k \cdot 3} (-1)\begin{vmatrix} 3 & -8 & -7 \\ l & m-2l & n-3l \\ 1 & 0 & 0 \end{vmatrix} =$$

$$=(-1)\left[-8n+24l-(-7m+14l)\right] = (-1)(-8n+\underline{24l}+7m-\underline{14l})$$

$$=(-1)(10l+7m-8n)=0 \quad \text{tj.} \quad 10l+7m-8n=0$$

$$\begin{array}{rl} 3l-2m-3n=0 & /\cdot 7 \\ 10l+7m-8n=0 & /\cdot 2 \\ \hline 21l-14m-21n=0 & \\ +\ 20l+14m-16n=0 & \\ \hline 41l-37n=0 & \end{array}$$

$41l = -37n$

$l = -\frac{37}{41}n$

$2m = 3l-3n$

$2m = -\frac{111}{41}n - \frac{123}{41}n$

$2m = \frac{-234}{41}n \quad /:2$

$m = \frac{-117}{41}n$

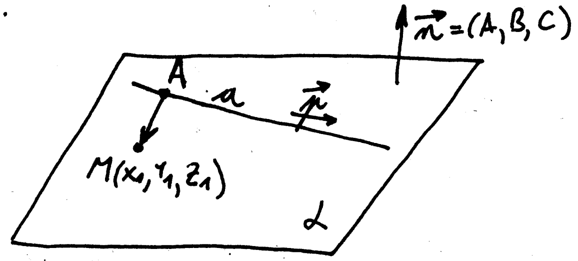
$\vec{p}_b = \left(-\frac{37}{41}n, \frac{-117}{41}n, n\right)$

Iz ovoga vidimo da za vektor pravca prave b mogu uzeti:

$\vec{p}_b = (-37,-117,41)$ $\quad b: \frac{x-3}{-37} = \frac{y+2}{-117} = \frac{z+4}{41}$ jednačina tražene prave

(#) Napisati jednačinu ravni koja sadrži tačku $M(1,-1,4)$ i pravu $\frac{x-1}{2}=\frac{y}{1}=\frac{z+1}{3}$.

Rj.



$\alpha: A(x-x_1)+B(y-y_1)+C(z-z_1)=0$

$a: \frac{x-1}{2}=\frac{y}{1}=\frac{z+1}{3}$

$\vec{p}=(l,m,n)=(2,1,3)$

$A\in a \quad A(1,0,-1)$

$A(1,0,-1)$
$M(1,-1,4) \quad \overrightarrow{AM}=(0,-1,5)$

$$\left.\begin{array}{l}\overrightarrow{AM}\perp\vec{n}\\ \vec{p}\perp\vec{n}\end{array}\right\} \Rightarrow \vec{n}\parallel \overrightarrow{AM}\times\vec{p}$$

$$\Downarrow$$

$$\vec{n}=k\cdot(\overrightarrow{AM}\times\vec{p}),\ k\in\mathbb{R}$$

$$\overrightarrow{AM}\times\vec{p}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & -1 & 5\\ 2 & 1 & 3\end{vmatrix}=\vec{i}(-8)-\vec{j}(-10)+\vec{k}\cdot 2=-8\vec{i}+10\vec{j}+2\vec{k}$$

$$\vec{n}=k(-8,10,2) \Rightarrow \vec{n}=(-4,5,1)$$

$M(1,-1,4)$
$\vec{n}=(-4,5,1)$

$$-4(x-1)+5(y+1)+1(z-4)=0$$

$$-4x+5y+z+4+5-4=0$$

$$-4x+5y+z+5=0$$ jednačina ravni koja sadrži datu tačku i datu pravu

(#)ˇ Date su ravni $\alpha: x+2y-z-5=0$ i $\beta: x-y+2z-2=0$. Nađi sve tačke na osi Oz koje su podjednako udaljene od ravni α i β.

(#)ˇ Dokazati da su prave $a: \frac{x+1}{3}=\frac{y-2}{2}=\frac{z+4}{1}$;
$b: \begin{cases} x-2y+z-3=0\\ 4x-5y-2z-3=0\end{cases}$ paralelne, pa zatim nađi jednačinu ravni koja ih sadrži.

(#) Kroz središte S duži određene tačkama $A(1,3,0)$ i $B(-3,7,2)$ postaviti pravu p paralelnu pravoj koja je zadana kao presjek ravni $\alpha: 6x-4y+z=16$ i $\beta: y+2z+1=0$.
Prava $g: \begin{cases} x=t+2 \\ y=t+2 \\ z=t+1 \end{cases}$, $t\in\mathbb{R}$ je zadana parametarski. Ispitati odnos između pravih p i g. Ukoliko nisu mimoilazne, napisati jednačinu ravni koja ih sadrži.

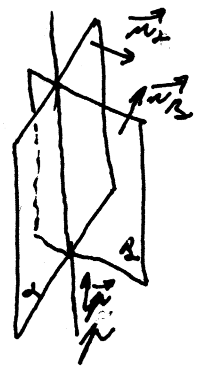
Rj. Nađimo središte S duži AB

$A(1,3,0)$
$B(-3,7,2)$ $\Rightarrow S(-1,5,1)$

$S\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

$\alpha: 6x-4y+z=16$
$\beta: \quad y+2z=-1$

Pronađimo koeficijent pravca prave koja je presjek ove dvije ravni



$\vec{n}_\alpha=(6,-4,1)$
$\vec{n}_\beta=(0,1,2)$

$\left.\begin{array}{l}\vec{p}\perp\vec{n}_\alpha \\ \vec{p}\perp\vec{n}_\beta\end{array}\right\} \Rightarrow \vec{p}\parallel \vec{n}_\alpha\times\vec{n}_\beta$
$\Downarrow$
$\exists k: \vec{p}=k(\vec{n}_\alpha\times\vec{n}_\beta)$

$\vec{n}_\alpha\times\vec{n}_\beta=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k} \\ 6 & -4 & 1 \\ 0 & 1 & 2\end{vmatrix}=$

$=-9\vec{i}-12\vec{j}+6\vec{k}$
$\Downarrow$
$\vec{p}=(-3,-4,2)$

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$
jednačina prave kroz jednu tačku

$$\frac{x+1}{-3}=\frac{y-5}{-4}=\frac{z-1}{2}$$ jednačina tražene prave p

$g: \begin{cases} x=t+2 \\ y=t+2 \\ z=t+1 \end{cases}$, $t\in\mathbb{R}$, ...(*) $g: \begin{cases} x-2=t \\ y-2=t \\ z-1=t \end{cases}$, $t\in\mathbb{R} \Rightarrow g: \frac{x-2}{1}=\frac{y-2}{1}=\frac{z-1}{1}$

Koeficijent pravca pave g je $\vec{p}_g=(1,1,1)$.
Prave p i g nisu paralelne (nije $\frac{l_1}{l_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2}$)

Pokušajmo naći presječnu tačku pravih p i g.

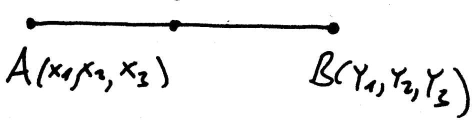
$p: \begin{cases} x=-3s-1 \\ y=-4s+5 \\ z=2s+1 \end{cases}$, $s\in\mathbb{R}$ ...(**)

(*) i (**) $\Rightarrow$
$-3s-1=t+2$ (1)
$-4s+5=t+2$ (2)
$2s+1=t+1$ (3)

(1)-(2):
$s-6=0$
$s=6 \Rightarrow$
$\Rightarrow t+2=-19$
$t=-\frac{19}{2}$
Kako ovaj t ne zadovoljava (3) sistem nema rješenja.
Prave p i g su mimoilazne.

(#) Kroz središte S duži određene tačkama $A(1,3,0)$ i $B(-3,7,2)$ postaviti pravu s paralelnu pravoj koja je zadana kao presjek ravni $\alpha: 6x-4y+z=16$ i $\beta: y+2z+1=0$.

Rj:



S sredina duži AB

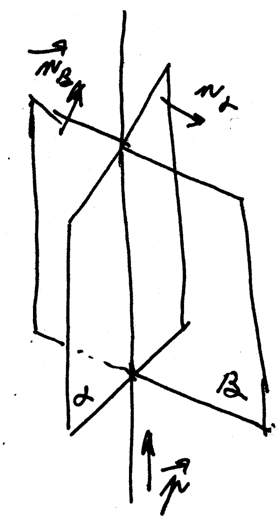
$$S\left(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}, \frac{x_3+y_3}{2}\right).$$

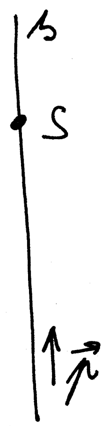
$A(1,3,0)$

$B(-3,7,2)$

$S(-1,5,1)$ sredina duži AB

$$s: \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$ jednačina prave kroz jednu tačku





$\vec{p} = (l, m, n)$

$$\left.\begin{array}{l}\vec{p} \perp \vec{n}_\alpha \\ \vec{p} \perp \vec{n}_\beta\end{array}\right\} \Rightarrow \vec{p} \parallel \vec{n}_\alpha \times \vec{n}_\beta$$

$\Downarrow$

$\exists k: \vec{p} = k(\vec{n}_\alpha \times \vec{n}_\beta)$

$\vec{n}_\alpha = (6,-4,1)$

$\vec{n}_\beta = (0,1,2)$

$$\vec{n}_\alpha \times \vec{n}_\beta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & -4 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -9\vec{i} - 12\vec{j} + 6\vec{k}$$
$$= (-9, -12, 6)$$

Pa za $\vec{p}$ možemo uzeti $\vec{p} = (3, 4, -2)$

$$s: \frac{x+1}{3} = \frac{y-5}{4} = \frac{z-1}{-2}$$ tražena jednačina prave.